6.1 Stability and Phase Plane

we focus on autonomous systems

$$\left.\begin{aligned}\frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y)\end{aligned}\right\}$$ F, G do not contain t explicitly (even though x, y depend on t)

these can be linear and homogeneous, for example

$$\left.\begin{aligned}\frac{dx}{dt} &= -2x-y \\ \frac{dy}{dt} &= -x-2y\end{aligned}\right\} \quad \vec{x}' = A\vec{x}$$

or linear and nonhomogeneous, for example

$$\frac{dx}{dt} = -2x-y+\underline{\underline{5}}$$

$$\frac{dy}{dt} = -x-2y+\underline{\underline{4}}$$

or nonlinear, for example

$$\frac{dx}{dt} = y^2$$

$$\frac{dy}{dt} = -x$$

the point (x,y) where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ is called a critical point or an equilibrium

all linear homogeneous systems $\vec{x}' = A\vec{x}$ have $(0,0)$ and only $(0,0)$ as the critical point.

for a nonhomogeneous linear system $\vec{x}' = A\vec{x} + \vec{g}$ where $\vec{g}$ is a constant vector will look like $\vec{x}' = A\vec{x}$ by translated to a different "origin" for example, $\left.\begin{aligned} x' &= -2x - y \\ y' &= -x - 2y \end{aligned}\right\}$ $\vec{x}' = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \vec{x}$

phase diagram is an improper nodal sink

$$\left.\begin{aligned} x' &= -2x - y + 5 \\ y' &= -x - 2y + 4 \end{aligned}\right\} \quad \vec{x}' = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

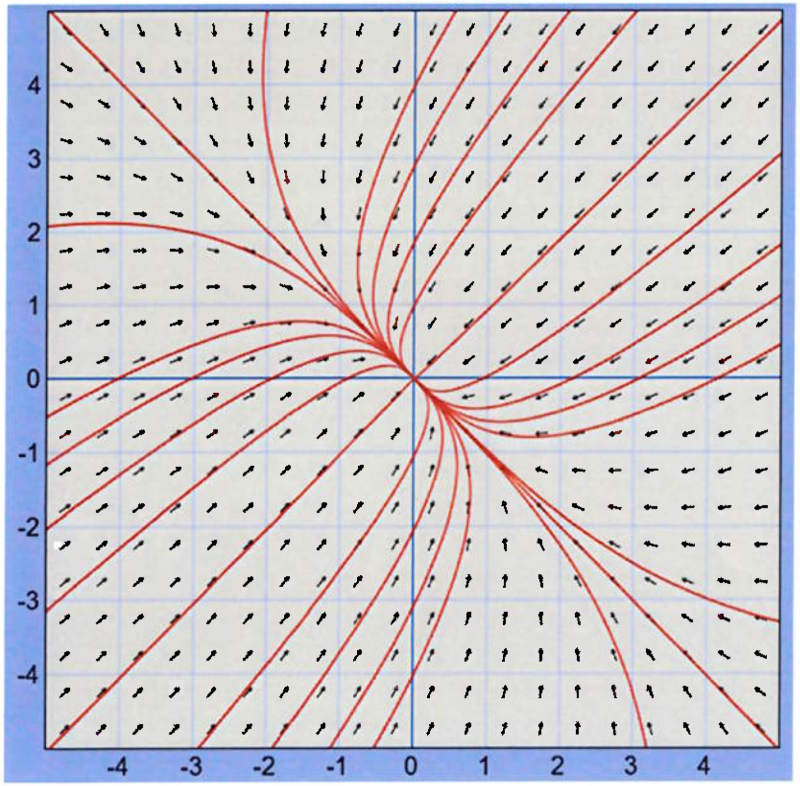
critical point : $-2x - y + 5 = 0$

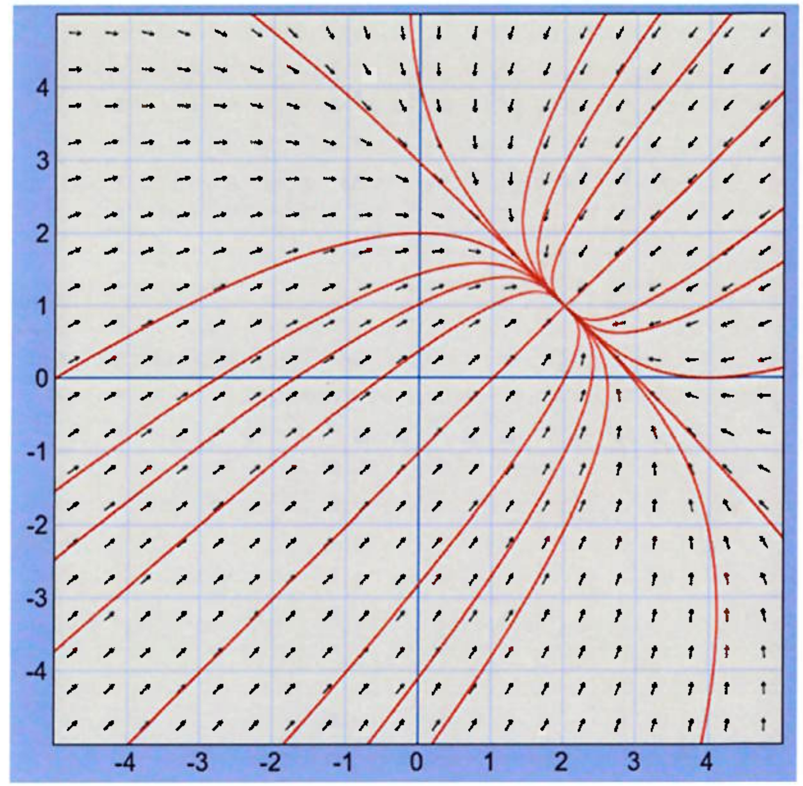
$-x - 2y + 4 = 0$

$\vdots$

$x = 2,\ y = 1$

so, whatever happens in $\vec{x}' = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \vec{x}$ stays the same but shifted to be centred at $(2, 1)$





$$x' = -2x - y$$
$$y' = -x - 2y$$

Critical point $(0,0)$

$$x' = -2x - y + 5$$
$$y' = -x - 2y + 4$$

Critical point $(2,1)$

Notice the phase portraits are identical but the system with constant nonhomogeneous term has the "origin" shifted a nonlinear system can have multiple critical points

for example, $x' = x(2-y) = 2x - xy$ nonlinear

$y' = y(x-3) = -3y + xy$

critical points : $x' = 0$ and $y' = 0$

$(0,0)$, $(3,2)$

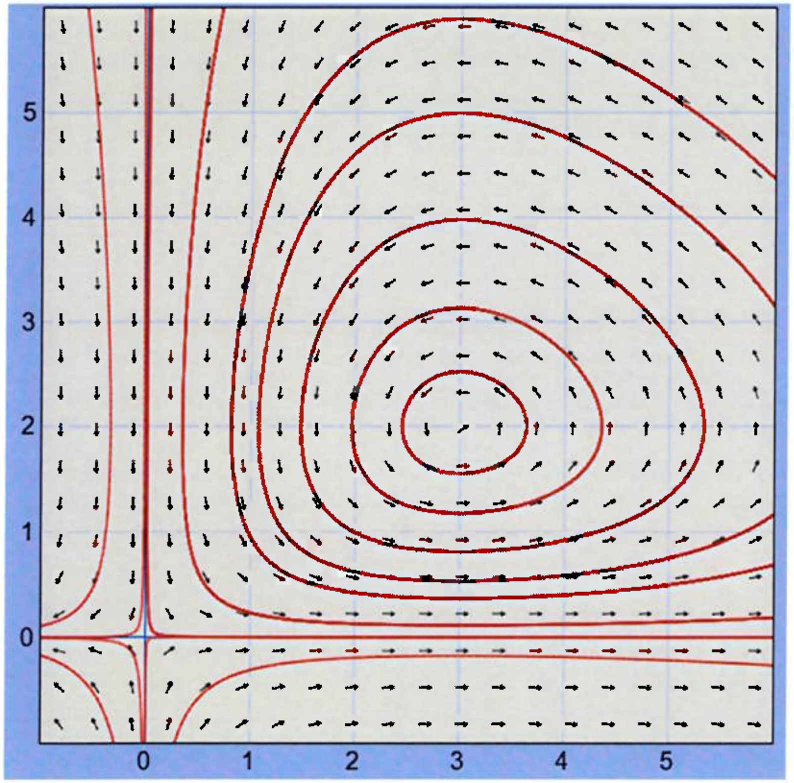
for many nonlinear systems, the solutions near each critical point resemble the solutions of a linear system but generally unpredictable away from critical pts.

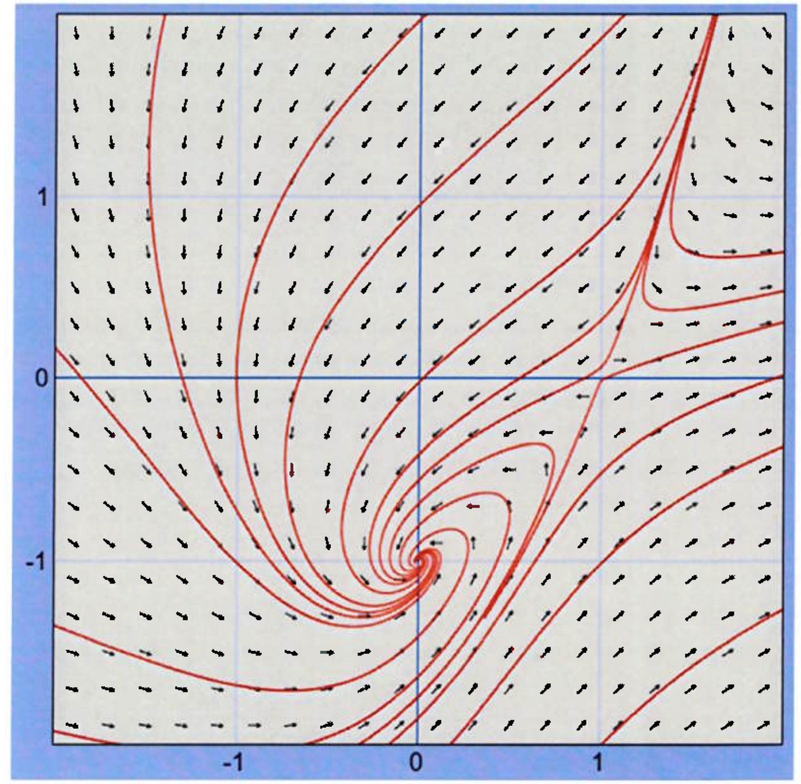
another example: $x' = x^2 - y - 1 \rightarrow x' = 0 \rightarrow y = x^2 - 1$

$y' = x - y - 1 \rightarrow y' = 0 \rightarrow x - (x^2 - 1) - 1 = 0$

$-x^2 + x = 0 \rightarrow x = 0,\ x = 1$

critical pts: $(0,-1)$, $(1,0)$





$$x' = x(2-y)$$
$$y' = y(x-3)$$

Critical points $(0,0)$, $(3,2)$

$$x' = x^2 - y - 1$$
$$y' = x - y - 1$$

Critical points $(0,-1)$, $(1,0)$

Notice each critical point resembles a center, improper/proper nodal source/sink saddle point, or spiral source/sink each critical pt is a solution that stays there

but what about nearby solutions?

if nearby solutions stay nearby, the crit. pt is <u>stable</u>

(center)

if nearby solutions fall into critical pt, the crit. pt. is

<u>asymptotically stable</u>

(sink of any sort)

if nearby solutions run away, the critical pt. is <u>unstable</u>

(source of any sort, saddle pt)

nonlinear systems are usually not easy to solve

Some can be solved: $$\frac{dx}{dt} = -y^2$$
$$\frac{dy}{dt} = x$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \boxed{\frac{x}{-y^2} = \frac{dy}{dx}}$$ separable

$$x\,dx = -y^2\,dy$$

$$\int x\,dx = \int -y^2\,dy$$

$$\frac{1}{2}x^2 = -\frac{1}{3}y^3 + C$$

$$\boxed{\frac{1}{3}y^3 + \frac{1}{2}x^2 = C}$$ each C gives one solution curve

but this is rarely possible w/ nonlinear sys in general